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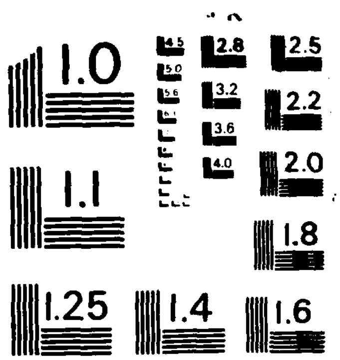

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MULTIPLE WIENER INTEGRALS AND NONLINEAR FUNCTIONALS OF A NUCLEAR SPACE VALUED WIENER PROCESS

by

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MULTIPLE WIENER INTEGRALS AND NONLINEAR FUNCTIONALS OF A NUCLEAR SPACE VALUED WIENER PROCESS*

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ABSTRACT

Let Φ' be the dual of a Countably Hilbert nuclear space and W_t be a Φ'-Wiener process. In this work we construct stochastic integrals and multiple Wiener integrals of operator valued processes with respect to W_t. The Wiener decomposition of the space of Φ'-valued nonlinear functionals of W_t is established. We also obtain multiple stochastic integral expansions and representations of Φ'-valued nonlinear functionals of W_t as operator valued stochastic integrals of Itô type.

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1. INTRODUCTION AND NOTATION

Stochastic processes taking values in duals of Countably Hilbert nuclear spaces have been considered in the works of Itô [9,10], Holley and Stroock [6], Dawson and Salehi [2], and Shiga and Shimizu [18] among others. In most of these papers, the nuclear space considered is $S'(\mathbb{R}^d)$, the space of tempered distributions. However, in several practical problems, e.g. those occurring in neurophysiology, it is not possible to fix in advance the space in which the stochastic processes take their values, as pointed out by Kallianpur and Wolpert [12]. Throughout this work we will assume that Φ is a Countably Hilbert nuclear space (CHNS) as defined in the work of the last named authors in the following manner: Suppose a strongly continuous semigroup $(T_t)_{t \geq 0}$ given on a Hilbert space H_0 (that can be taken as $H_0 = L^2(X, d\Gamma)$ for some σ-finite measure space (X, A, Γ)). The semigroup $(T_t)_{t \geq 0}$ usually describes the evolutionary phenomenon being studied, such as the behavior of the voltage potential of a neuron (see [12]). Suppose that the strongly continuous and self adjoint semigroup $(T_t)_{t \geq 0}$ satisfies the following two conditions:

(i) The resolvent $R_\alpha = \int_0^\infty e^{-\alpha t} T_t \, dt$ is compact for each $\alpha > 0$.

(ii) For some $r_1 > 0$ $(R_\alpha)^{r_1}$ is a Hilbert-Schmidt operator.

By the Hille-Yosida theorem (T_t) has a negative definite infinitesimal generator $-L$ and H_o admits a complete orthonormal set $\{\phi_j\}_{j \geq 1}$ of eigenvectores of L with eigenvalues $0 < \lambda_1 < \lambda_2 < \ldots$ satisfying

$$\sum_{j=1}^{\infty} (\alpha + \lambda_j)^{-2r_1} < \infty \quad (r_1 > 0) \quad . \tag{1.1}$$

Set

$$\theta_1 = \sum_{j=1}^{\infty} (1 + \lambda_j)^{-2r_1} \quad . \tag{1.2}$$

Denote by $<\cdot,\cdot>_o$ the inner product in H_o and let

$$\Phi = \{\phi \in H_o : \sum_{j=1}^{\infty} <\phi,\phi_j>_o^2 (1 + \lambda_j)^{2r} < \infty \text{ for all } r \in \mathbb{R}\} \quad . \tag{1.3}$$

For each $r \in \mathbb{R}$ define an inner product $<\cdot,\cdot>_r$ and norm $\|\cdot\|$ on Φ by

$$<\phi,\psi>_r = \sum_{j=1}^{\infty} <\phi,\phi_j>_o <\psi,\phi_j>_o (1 + \lambda_j)^{2r} \tag{1.4}$$

$$\|\phi\|_r^2 = <\phi,\phi>_r \tag{1.5}$$

and let H_r be the Hilbert space completion of Φ in the inner product $<\cdot,\cdot>_r$. Then Φ with the Frechet topology determined

by the family $\{\|\cdot\|_r\}_{r \in \mathbb{R}}$ of Hilbertian norms is a Countably Hilbert nuclear space. Let $\Phi' = \bigcup_r H_r$ with the inductive limit topology. Then Φ' is identified with the dual space (in the weak topology) to Φ. The following are straight forward consequences (see [12]):

i) $\Phi \subset H_s \subset H_r \subset \Phi'$, $\|\phi\|_r \leq \|\phi\|_s$ for $r < s$.

ii) The injection of H_s into H_r is a Hilbert-Schmidt map if $s > r + r_1$.

iii) Let $H_{-r} = H_r'$ denote the strong dual of the Hilbert space H_r. Then H_{-r} and H_r are in duality under the pairing

$$\xi[\phi] = \sum_{j=1}^{\infty} \langle \xi, \phi_j \rangle_{-r} \langle \phi, \phi_j \rangle_r \quad \xi \in H_{-r},\ \phi \in H_r . \tag{1.6}$$

iv) Finite linear combinations of $\{\phi_j\}$ are dense in Φ and in every H_r; moreover $\{\phi_j\}_{j \geq 1}$ is an orthogonal system in each H_r, and then $\{(1 + \lambda_j)^{-r}\phi_j\}_{j \geq 1}$ is a CONS for H_r.

The spaces $S(\mathbb{R}^d)$ and $S(\mathbb{Z}^d)$ of rapidly decreasing functions on $\mathbb{R}^d$ and rapidly decreasing sequences in $\mathbb{Z}^d$ respectively, may be obtained in the above framework (see [1] and [17]).

Throughout this work we will assume that (Ω, F, P) is a complete probability space on which all Φ'-valued stochastic processes will be defined.

Definition 1.1. A sample continuous Φ'-valued stochastic process $W = (W_t)_{t \geq 0}$ is called a (centered) Φ-valued Wiener process with covariance $Q(\cdot,\cdot)$ if

a) $W_o \equiv 0$.

b) W_t has independent increments.

c) For each $\phi \in \Phi$ and $t \geq 0$

$$E(e^{iW_t[\phi]}) = \exp(-t/2\ Q(\phi,\phi))$$

where $Q(\cdot,\cdot)$ is a continuous positive definite bilinear (c.p.d. b.) form on $\Phi \times \Phi$.

It is easily seen that the system $\{W_t[\phi] : \phi \in \Phi, t \geq 0\}$ is a Gaussian system of random variables and that if ϕ, $\psi \in \Phi$, then the real valued processes $W_t[\phi]$ and $W_t[\psi]$ are independent on non-overlapping increments. Moreover for s, $t \in \mathbb{R}_+$

$$E(W_s[\phi] W_t[\psi]) = \min(s,t)\ Q(\phi,\psi). \tag{1.7}$$

If $Q(\cdot,\cdot) = \langle\cdot,\cdot\rangle_o$, following Itô [9], W_t may be called a Standard Φ'-valued Wiener process.

Using the Nuclear theorem ([5]) one can show that if Q is a c.p.d.b. form on $\Phi \times \Phi$, then there exist $r_2 > 0$ and $\theta_2 > 0$ such that for $\phi \in \Phi$

$$Q(\phi,\phi) \leq \theta_2 \|\phi\|^2_{r_2} . \tag{1.8}$$

A regularization technique, as that used in Itô [10], shows that given a c.p.d.b. form Q on $\Phi \times \Phi$ there exists a Φ'-valued Wiener process W_t with covariance Q. Moreover (see [17, Th. 4.1.1]) W_t has an H_{-q}-valued continuous version for $q \geq r_1 + r_2$. Let H_Q be the completion of Φ w.r.t. $Q(\cdot,\cdot)$. Then to every Φ'-valued Wiener process we can associate a Rigged Hilbert space ([5])

$$\Phi \subset H_s \subset H_Q \equiv H_Q' \subset H_{-s} \subset \Phi' \qquad s \geq r_2 . \tag{1.9}$$

Examples of Φ'-valued Wiener processes arising in different situations are presented in [17].

In Section 2 of this work we present "weak" stochastic integrals similar to the case of a cylindrical Brownian motion as presented in Yor [21]. We consider real valued and Φ'-valued stochastic integrals including operator valued processes as integrands. It turns out that these integrals are the ones useful in representing nonlinear functionals of W_t.

In Section 3 we present real valued and Φ'-valued multiple Wiener integrals w.r.t. W_t including operator valued integrands. Our method leads to consider multiple Wiener integrals with dependent integrators for real valued integrands of the type considered in the recent works [3], [4] and [17].

In Section 4 we obtain the Wiener decomposition of the space $L^2(\Omega;\Phi')$ of Φ'-valued nonlinear functionals of W_t. Furthermore we consider multiple Wiener integral expansions and stochastic integral representations for elements in $L^2(\Omega;\Phi')$, as well as representation theorems for Φ'-valued square integrable martingales. These results are the Φ'-valued analog of the corresponding results for nonlinear functionals of a real valued Wiener process, as presented for example in Kallianpur [11, Ch. 6].

An important role in this work is played by a Baire category argument, first used in the study of nuclear space valued stochastic processes in Mitoma [15].

This work is motivated by the need for developing techniques for the study of nonlinear models which describe the neurophysiological applications presented in Kallianpur and Wolpert [12].

2. STOCHASTIC INTEGRALS

Stochastic integrals with respect to $S'(\mathbb{R}^d)$-valued Wiener processes and E'-valued (E is a CHNS) processes have been discussed in Itô [9,10] and Mitoma [15] respectively. They propose to use the theory of stochastic integration on Hilbert spaces, as presented for example in Kunita [13] or Kuo [14], to construct stochastic integrals for the H_{-q} valued Wiener process W_t. In this section we present weak stochastic integrals similar to the case of a cylindrical Brownian motion (c.B.m.). However, we do not work with a c.B.m. but rather with a Φ'-Wiener process (a true process) with an H_{-q}-valued continuous version for $q \geq r_1 + r_2$. Secondly, if $\{e_k\}$ is any CONS in H_q then $\{W_t[e_k]\}_{k \geq 1}$ is not necessarily a system of independent random variables (see (1.7)), as it would be required in the case of a c.B.m.. Moreover, we do not assume that the common orthogonal system in H_r $r \geq 0$ $\{\phi_j\}_{j \geq 1}$ (the eigenvectors of the infinitesimal generator L) diagonalizes Q. The case when $Q(\cdot,\cdot) = \langle\cdot,\cdot\rangle_o$, and then $\{\phi_j\}_{j \geq 1}$ diagonalizes Q, has been considered by Daletskii [1] and Miyahara [16]. Nevertheless, the nuclearity of the space $(\Phi, \|\cdot\|_r\ r \geq 0)$ enables us to construct real valued and Φ'-valued stochastic integrals. We also make extensive use of the c.p.d.b. form Q and its associated Rigged Hilbert space (1.9).

Let $F_t = F_t^W = \sigma(W_s : 0 \le s \le t)$ with F_o containing all P-null sets of F and let $F^W = F_\infty^W$.

Real Valued Stochastic Integrals.- Let K be a real separable Hilbert space. A function $f: [0,\infty) \times \Omega \to K$ is said to belong to the class $M(W,K)$ if f is an F_t-adapted measurable (non-anticipative) function on $\mathbb{R}_+ \times \Omega$ to K such that for each $t > 0$

$$\int_o^t E\|(f(s)\|_K^2 \; ds < \infty .$$

The special classes we will be concerned with are $M_q = M(W,H_q)$, $q \ge r_1 + r_2$ and $M_Q = M(W,H_Q)$.

We first define stochastic integrals for elements in M_q.

Definition 2.1 Let $q \ge r_1 + r_2$. For $g \in M_q$ and $t > 0$ define the real valued stochastic integral $\int_o^t \langle g_s, dW_s\rangle_q$ as

$$\int_o^t \langle g_s, dW_s\rangle_q = \sum_{i=1}^{\infty} \int_o^t \langle g_s, e_i\rangle_q \; dW_s[e_i] \tag{2.1}$$

where $\{e_i\}_{i \ge 1}$ is a CONS for H_q and the integrals on the right hand side of (2.1) are ordinary Itô integrals.

Proposition 2.1 Let $g \in M_q$ $q \ge r_1 + r_2$. Then the integral (2.1) is a well defined element in $L^2(\Omega, F^W, P)$. If

$q_1 \geq r_1 + r_2$ and $g \in M_{q_1}$ then this integral is independent of q or q_1. Moreover the following properties are satisfied for $f, g \in M_q$.

a) For $a, b \in \mathbb{R}$ and $t > 0$

$$\int_0^t \langle af_s + bg_s, dW_s \rangle_q = a \int_0^t \langle f_s, dW_s \rangle_q + b \int_0^t \langle g_s, dW_s \rangle_q \text{ a.s.}.$$

b) $E(\int_0^t \langle g_s, dW_s \rangle_q) = 0 \quad t > 0$.

c) $E(\int_0^{t_1} \langle g_s, dW_s \rangle_q \int_0^{t_2} \langle f_s, dW_s \rangle_q) = E \int_0^{t_1 \wedge t_2} Q(f_s, g_s) ds$

$t_1 > 0,\ t_2 > 0$.

d) $E(\int_0^t \langle f_s, dW_s \rangle_q)^2 = E \int_0^t Q(f_s, f_s) ds \leq E \int_0^t \| f_s \|_q^2 \, ds < \infty$.

e) The real valued process $\{\int_0^t \langle g_s, dW_s \rangle_q\}_{t \geq 0}$ is an F_t-martingale with associated increasing process $E\int_0^t Q(g_s, g_s) ds$.

<u>Proof</u> We first prove that for $t > 0$ $\int_0^t \langle g_s, dW_s \rangle_q$ is a well defined element in $L^2(\Omega, F^W, P)$. Let $\{e_i\}_{i \geq 1}$ be any CONS for H_q $q \geq r_1 + r_2$. Then for each $t > 0$

$$g(t,\omega) = \sum_{j=1}^{\infty} \langle g_t(\omega), e_j\rangle_q \; e_j$$

and for $n, m \geq 1$, using the fact that for $\phi, \psi \in H_q$ the cross predictable quadratic variation of $W_t[\phi]$ and $W_t[\psi]$ is

$$\langle W[\phi], W[\psi]\rangle_t = t\, Q(\phi,\psi) \qquad t \geq 0 ,$$

we obtain that

$$E\Big(\sum_{j=m}^{n} \int_0^t \langle g_s, e_j\rangle_q \, dW_s[e_j]\Big)^2 =$$

$$\sum_{j=m}^{n} \sum_{k=m}^{n} E \int_0^t \langle g_s, e_j\rangle_q \langle g_s, e_k\rangle_q \, Q(e_j, e_k) ds \; .$$

Then since Q is a c.p.d.b. form, using (1.8) we have that

$$E\Big(\sum_{j=m}^{n} \int_0^t \langle g_s, e_j\rangle_q \, dW_s[e_j]\Big)^2$$

$$= E \int_0^t Q\Big(\sum_{j=m}^{n} \langle g_s, e_j\rangle_q \, e_j, \; \sum_{j=m}^{n} \langle g_s, e_j\rangle_q \, e_j\Big) ds$$

$$\leq \theta_2 \, E \int_0^t \Big\| \sum_{j=m}^{n} \langle g_s, e_j\rangle_q \, e_j \Big\|_q^2 \, ds$$

$$= \theta_2 \, E \int_0^t \Big(\sum_{j=m}^{n} \langle g_s, e_j\rangle_q^2\Big) ds \to 0 \quad \text{as} \quad n,m \to \infty$$

since $g \in M_q$.

Thus $\int_0^t \langle g_s, dW_s\rangle_q$ is an element of $L^2(\Omega, F^W, P)$ defined as the $L^2(\Omega)$-limit of the Cauchy sequence

$$\{\sum_{j=1}^n \int_0^t \langle g_s, e_j\rangle_q \; dW_s[e_j]\}_{n \geq 1} .$$

The next argument will also show that (2.1) is independent of the CONS $\{e_j\}_{j \geq 1}$ in H_q. Let $q_1 \geq r_1 + r_2$ $q_1 > q$ and $\{\psi_j\}_{j \geq 1}$ be a CONS for H_{q_1}. Then $\|\cdot\|_{r_2} \leq \|\cdot\|_q \leq \|\cdot\|_{q_1}$, $H_{q_1} \subset H_{q_2} \subset H_{r_2}$ and W_t has an H_{-q_1}-valued continuous version. Hence if $g \in M_q \cap M_{q_1}$

$$E\left(\sum_{j=1}^n \int_0^t \langle g_s, e_j\rangle_q \; dW_s[e_j] - \sum_{j=1}^n \int_0^t \langle g_s, \psi_j\rangle_{q_1} dW_s[\psi_j]\right)^2 =$$

$$E\left(\int_0^t Q\left(\sum_{j=1}^n \langle g_s, e_j\rangle_q \, e_j - \sum_{j=1}^n \langle g_s, \psi_j\rangle_{q_1} \psi_j, \sum_{j=1}^n \langle g_s, e_j\rangle_q \, e_j - \right.\right.$$

$$\left.\left.\sum_{j=1}^n \langle g_s, \psi_j\rangle_{q_1} \psi_j\right) ds\right)$$

$$\leq \theta_2 E \int_0^t \left\| \sum_{j=1}^n \langle g_s, \psi_j\rangle_{q_1} \psi_j - \sum_{j=1}^n \langle g_s, e_j\rangle_q \, e_j \right\|_q^2 ds$$

$\to 0$ as $n \to \infty$ by dominated convergence theorem.

Thus the integral (2.1) is independent of q and q_1.

The proofs of (a) - (e) follow using the usual techniques. For details see [17].

Q.E.D.

As in the case of a real valued Wiener process, under an additional condition we can define the integral $\int_0^\infty \langle f_s, dW_s\rangle_q$.

<u>Definition 2.2</u> For $q \geq r_1 + r_2$ let $f:[0,\infty) \times \Omega \to H_q$ be a non-anticipative H_q-valued process such that

$$\int_0^\infty E\| f(s) \|_q^2 \, ds < \infty. \tag{2.2}$$

Define $\int_0^\infty \langle f_s, dW_s\rangle_q$ as the mean square limit of $\int_0^t \langle f_s, dW_s\rangle_q$ as $t \to \infty$. Then this integral is well defined and has the properties (a) - (d) of Proposition 2.1 writing ∞ instead of t. Moreover, for all $t > 0$

$$E\left(\int_0^\infty \langle f_s, dW_s\rangle_q \,|\, F_t\right) = \int_0^t \langle f_s, dW_s\rangle_q \qquad \text{a.s.}$$

and $(\int_0^t \langle f_s, dW_s\rangle_q, F_t)_{t \geq 0}$ is a square integrable martingale with increasing process $E \int_0^t Q(f_s, f_s)ds$ and a continuous version on $\mathbb{R}_+$.

For $f \in M_Q$ a stochastic integral of the form (2.1) cannot be

defined since W_t is not an H_Q-valued process. However, we are still able to define a stochastic integral with the help of the following lemma.

<u>Lemma 2.1</u> Let $q > r_1 + r_2$ and $f \in M_Q$. Then there exists a sequence $\{f_n\}_{n \geq 1}$ in M_q such that for each $t > 0$

$$\int_0^t E\| f(s) - f_n(s) \|_Q^2 \, ds \to 0 \quad \text{as} \quad n \to \infty.$$

<u>Proof</u> Let $\{e_i\}_{i \geq 1}$ be a CONS for H_Q and let P_n be the orthogonal projector onto the span of $\{e_1,\ldots,e_n\}$. For each $t > 0$ by monotone convergence theorem

$$\int_0^t E\| f_s \|_Q^2 \, ds = \sum_{j=1}^{\infty} \int_0^t E(<f_s, e_j>_Q^2) \, ds$$

and hence for each $t > 0$

$$\int_0^t E\| P_n f_s - f_s \|_Q^2 \, ds = \sum_{j=n+1}^{\infty} \int_0^t E(<f_s, e_j>_Q^2) \, ds \to 0$$

as $n \to \infty$.

Next for all $n \geq 1$ there exists a sequence $(\beta_k^n)_{k \geq 1}$ of non-anticipative step processes with values in the range of P_n (this is the finite dimensional case, see for example Lemma 4.3.2 in Strook and Varadhan [19] or Lemma 1.1 in Ikeda and

Watanabe [7]) such that for each $t > 0$

$$\int_0^t E\| \beta_k^n(s) - P_n f_s \|_Q^2 \, ds < \frac{1}{k} \qquad k = 1,2,\ldots .$$

Define the H_Q-valued step process

$$\alpha_n(t)(\omega) = \beta_n^n(t)(\omega) \qquad 0 \leq t < \infty \qquad \omega \in \Omega \qquad n \geq 1.$$

Then for all $t > 0$

$$\int_0^t E\| \alpha_n(s) - f_s \|_Q^2 \, ds \leq \int_0^t E\| \alpha_n(s) - P_n f_s \|_Q^2 \, ds$$

$$+ \int_0^t E\| P_n f_s - f_s \|_Q^2 \, ds \leq \frac{1}{n} + \int_0^t E\| P_n f_s - f_s \|_Q^2 \, ds \to 0$$

as $n \to \infty$.

Thus we have shown that if $f \in M_Q$, for all $\varepsilon > 0$ there exists an H_Q-valued step process $\alpha(t,\omega)$ such that for each $t > 0$

$$\int_0^t E\| \alpha(s) - f(s) \|_Q^2 \, ds < \varepsilon/4$$

where

$$\alpha(s,\omega) = \alpha_{t_j}(\omega) \quad \text{a.s.} \quad t_j < s < t_{j+1} \quad j = 0,\dots,n-1$$

$$= \alpha_{t_n}(\omega) \quad \text{a.s.} \quad s > t_n$$

where $0 = t_0 < t_1 < \dots < t_n < \infty$ and each α_{t_j} takes values in a finite dimensional subspace B_j of H_Q, it is F_{t_j}-measurable and $E\|\alpha_{t_j}\|_Q^2 < \infty$ for $j = 1,\dots,n$.
Next for each $j=1,\dots,n$ let $\{e_1^j,\dots,e_{k_j}^j\}$ be an orthogonal basis for B_j.
Since H_q is dense in H_Q we can choose $\{\psi_1^j,\dots,\psi_{k_j}^j\}$ such that $\psi_\ell^j \in H_q$ and

$$\|\psi_\ell^j - e_\ell^j\|_Q^2 < \frac{\varepsilon}{2k_j(t_{j+1}-t_j)E\|\alpha_{t_j}\|_Q^2} \quad \ell = 1,\dots,k_j \ .$$

Each α_{t_j} can be written as

$$\alpha_{t_j}(\omega) = a_1^j(\omega)\ e_1^j + \dots + a_{k_j}^j(\omega)\ e_{k_j}^j$$

where

$$E\|\alpha_{t_j}\|_Q^2 = E((a_1^j)^2 + \dots + (a_{k_j}^j)^2) < \infty.$$

Define

$$\alpha_{t_j}^*(\omega) = a_1^j(\omega)\ \psi_1^j + \dots + a_{k_j}^j(\omega)\ \psi_{k_j}^j$$

then

$$E\| \alpha^*_{t_j} \|^2_q \le E((a^j_1)^2 + \dots + (a^j_{k_j})^2) \sum_{i=1}^{k_j} \|\psi^j_i\|^2_q < \infty$$

and

$$E\| \alpha_{t_j} - \alpha^*_{t_j}\|^2_Q = E\| \sum_{i=1}^{k_j} a^j_i(e^j_i - \psi^j_i)\|^2_Q$$

$$\le E(\sum_{i=1}^{k_j} |a^j_i| \; \| e^j_i - \psi^j_i \|_Q)^2$$

$$\le \{E(\sum_{i=1}^{k_j} (a^j_i)^2)\}\{ \sum_{i=1}^{k_j} \| e^j_i - \psi^j_i \|^2_Q\} < \frac{\varepsilon}{2(t_{j+1} - t_j)} .$$

Finally define

$$\alpha^*(s,\omega) = \begin{cases} \alpha^*_{t_j}(\omega) & t_j \le s < t_{j+1} \qquad j = 1,\dots,n-1 \\ \alpha^*_{t_n}(\omega) & s \ge t_n \end{cases}$$

which is an element of M_q. Then for each $f \in M_Q$ and $\varepsilon > 0$ there exists $\alpha^* \in M_q$ such that for each $t > 0$

$$\int_0^t E\| \alpha^*(t) - f(t)\|^2_Q \, dt < \epsilon$$

and the existence of the required sequence follows.

Q.E.D.

Definition 2.3 Let $f \in M_Q$, then from Lemma 2.1 there exists a sequence of functions $\{f_n\}_{n \geq 1}$ in M_q for $q \geq r_1 + r_2$, such that for each $t > 0$

$$\int_o^t E\| f(s) - f_n(s) \|_Q^2 \, ds \to 0 \quad \text{as} \quad n \to \infty .$$

Then by Proposition 2.1 (d) for each $t > 0$

$$E\left(\int_o^t \langle f_n(s) - f_m(s), dW_s \rangle_q\right)^2 = \int_o^t E\| f_n(s) - f_m(s) \|_Q^2 \, ds \to 0$$

$n, m \to \infty$.

Define for each $t > 0$ the stochastic integral $\int_o^t \langle f_s, dW_s \rangle_Q$ as the $L^2(\Omega)$-limit of the Cauchy sequence $\{\int_o^t \langle f_n(s), dW_s \rangle_q\}_{n \geq 1}$. If in addition f is such that

$$\int_o^\infty E\| f_s \|_Q^2 \, ds < \infty$$

then the stochastic integral $\int_o^\infty \langle f_s, dW_s \rangle_Q$ is defined as the mean square limit of $\int_o^t \langle f_s, dW_s \rangle_Q$ as $t \to \infty$.

For the sake of completness the main properties of the above integral are summarized in the following:

<u>Proposition 2.2</u> Let $f, g \in M_Q$. Then

a) If $a, b \in \mathbb{R}$ and $t > 0$

$$\int_0^t \langle af_s+bg_s, dW_s\rangle_Q = a\int_0^t \langle f_s, dW_s\rangle_Q + b\int_0^t \langle g_s, dW_s\rangle_Q \quad \text{a.s.}$$

b) $E(\int_0^t \langle f_s, dW_s\rangle_Q) = 0 \qquad \text{all } t > 0$.

c) $E(\int_0^{t_1} \langle f_s, dW_s\rangle_Q \int_0^{t_2} \langle g_s, dW_s\rangle_Q) = E\int_0^{t_1 \wedge t_2} \langle f_s, g_s\rangle_Q \, ds$

$t_1, t_2 > 0$.

d) $E(\int_0^t \langle f_s, dW_s\rangle_Q)^2 = E\int_0^t \| f_s\|_Q^2 \, ds < \infty$.

e) If $E\int_0^\infty \|f_s\|_Q^2 \, ds < \infty$ then $\{\int_0^t \langle f_s, dW_s\rangle_Q, F_t\}_{t \geq 0}$

is a square integrable martingale with corresponding increasing process $\int_0^t E\|f_s\|_Q^2 \, ds$ and a continuous modification on $\mathbb{R}_+$. Moreover for $t > 0$

$$E(\int_0^\infty \langle f_s, dW_s\rangle_Q | F_t) = \int_0^t \langle f_s, dW_s\rangle_Q \quad \text{a.s.}$$

and

$$E(\int_0^\infty \langle f_s, dW_s\rangle_Q)^2 = E\int_0^\infty \| f_s\|_Q^2 \, ds.$$

The proof follows by the above definition and Proposition 2.1.

<u>Φ'-Valued Stochastic Integrals</u>.- Let $L(\Phi',\Phi')$ denote the class of continuous linear operators from Φ' to Φ'. A function $f:[0,\infty) \times \Omega \to L(\Phi',\Phi')$ is said to belong to the class $O_Q(\Phi',\Phi')$ if f is an F_t-adapted measurable (non-anticipative) function on $[0,\infty) \times \Omega$ to $L(\Phi',\Phi')$ such that for each $t > 0$

$$E \int_0^t Q(f_s^*(\phi),f_s^*(\phi))\, ds < \infty \quad \forall\, \phi \in \Phi \tag{2.3}$$

where $f_s^*: \Phi \to \Phi$ is the adjoint of f_s.

<u>Lemma 2.2</u> Let $f \in O_Q(\Phi',\Phi')$. Then for each $t > 0$ there exists $q(t,f) \geq r_1 + r_2$ such that

$$E \int_0^t \| f_s^* \|^2_{\sigma_2(H_{q(t,f)},H_Q)} ds = E \int_0^t \| f_s \|^2_{\sigma_2(H_Q,H_{-q(t,f)})} ds < \infty \tag{2.4}$$

where $\sigma_2(H_{q(t,f)},H_Q)$ denotes the Hilbert space of Hilbert-Schmidt operators from $H_{q(t,f)}$ to H_Q.

<u>Proof</u> For each $t > 0$ and $\phi \in \Phi$ let

$$V_t^2(\phi) = E \int_0^t Q(f_s^*(\phi),f_s^*(\phi))ds. \tag{2.5}$$

Then since $f \in \mathcal{O}_Q(\Phi',\Phi')$ for each $t > 0$ $V_t^2(\phi) < \infty \quad \forall \phi \in \Phi$.

Let $\phi_n \to \phi$ in Φ, then since $f_s^* \in L(\Phi,\Phi)$ and Q is Φ-continuous, using Fatou's lemma we have that

$$V_t(\phi) = \{E \int_o^t \liminf Q(f_s^*(\phi_n), f_s^*(\phi_n))ds\}^{1/2} \leq \liminf V_t(\phi_n)$$

which shows that V_t is a lower semicontinuous function on Φ. Then using Lemma I.2.3 in [20, Page 386] $V_t(\phi)$ is a continuous function on Φ and there exist $r(t,f) > 0$ and $\theta(t,f) > 0$ such that

$$V_t^2(\phi) \leq \theta(t,f)^2 \|\phi\|^2_{r(t,f)} \qquad \forall \phi \in \Phi . \tag{2.6}$$

Next let $\{\phi_j\}_{j \geq 1}$ and $\{\lambda_j\}_{j \geq 1}$ be as in Section 1. Choose $q(t,f) \geq r(t,f) + r_1$ and write $\tilde{\phi}_j = (1 + \lambda_j)^{-q(t,f)}\phi_j \quad j \geq 1$. Then $\{\tilde{\phi}_j\}_{j \geq 1}$ is a CONS for $H_{q(t,f)}$ and using (2.5) and (2.6) we have that

$$E \int_o^t (\sum_{j=1}^{\infty} Q(f_s^*(\tilde{\phi}_j), f_s^*(\tilde{\phi}_j)))ds = \sum_{j=1}^{\infty} V_t^2(\tilde{\phi}_j)$$

$$\leq \theta(t,f)^2 \sum_{j=1}^{\infty} \|\tilde{\phi}_j\|^2_{r(t,f)} = \theta(t,f)^2 \sum_{j=1}^{\infty} (1 + \lambda_j)^{-2(q(t,f)-r(t,f))}$$

$$\leq \theta(t,f)^2 \theta_1 < \infty,$$

and (2.4) is proved.

Proposition 2.3 Let $f \in \mathcal{O}_Q(\Phi',\Phi')$. Then for each $t > 0$ there exists a Φ'-valued element $Y_t(f)$ such that

$$Y_t(f)[\phi] = \int_0^t \langle f_s^*(\phi), dW_s\rangle_Q \quad \text{a.s.} \quad \forall\, \phi \in \Phi \tag{2.7}$$

where the RHS of (2.7) is the stochastic integral of Definition 2.3. Moreover, for each $T_0 > 0$ there exists a positive integer $q(T_0,f)$ such that $Y_t(f) \in H_{-q(T_0,f)}$ a.s. for $0 \leq t \leq T_0$. $Y_t(f)$ is called the Φ'-valued stochastic integral of f w.r.t. W and sometimes we will denote it by

$$Y_t(f) = \int_0^t f_s dW_s. \tag{2.8}$$

Proof Using the notation of the proof of Lemma 2.2, define

$$Y_t(f)[\tilde{\phi}_j] = \int_0^t \langle f_s^*(\tilde{\phi}_j), dW_s\rangle_Q \qquad j \geq 1 .$$

Then by Proposition 2.2 (d), (2.5) and (2.6)

$$E\Big(\sum_{j=1}^{\infty} (Y_t(f)[\tilde{\phi}_j])^2\Big) = \sum_{j=1}^{\infty} E(Y_t(f)[\tilde{\phi}_j])^2 = \sum_{j=1}^{\infty} V_t^2(\tilde{\phi}_j)$$

$$\leq \theta(t,f)^2 \sum_{j=1}^{\infty} \| \tilde{\phi}_j \|^2_{r(t,f)} \leq \theta(t,f)^2\, \theta_1 < \infty.$$

Thus $\sum_{j=1}^{\infty} (Y_t(f)[\tilde{\phi}_j])^2 < \infty$ a.s.. Let $\Omega_1 = \{\omega: \sum_{j=1}^{\infty} (Y_t(f)[\tilde{\phi}_j](\omega))^2 < \infty\}$ then $P(\Omega_1) = 1$.

Let $\{\psi_j\}_{j \geq 1}$ be the CONS for $H_{-q(t,f)}$ dual to $\{\tilde{\phi}_j\}_{j \geq 1}$ and define

$$\tilde{Y}_t(f)(\omega) = \begin{cases} \sum_{j=1}^{\infty} Y_t(f)[\tilde{\phi}_j](\omega)\psi_j & \omega \in \Omega_1 \\ 0 & \omega \notin \Omega_1 \end{cases} .$$

Then for each $t > 0$ $\tilde{Y}_t(f) \in H_{-q(t,f)}$ a.s. for $q(t,f) > r(t,f) + r_1$ and therefore $\tilde{Y}_t(f) \in \Phi'$ a.s. .

It remains to prove that $\tilde{Y}_t$ satisties (2.7). Let $t > 0$ and $\phi \in \Phi$, then $\phi \in H_{q(t,f)}$ and

$$\phi = \lim_{n\to\infty} \sum_{j=1}^{n} \langle\phi,\tilde{\phi}_j\rangle_{q(t,f)}\tilde{\phi}_j \quad (\text{limit in } H_{q(t,f)})$$

and therefore $V_t(\sum_{j=1}^{n} \langle\phi,\tilde{\phi}_j\rangle_{q(t,f)}\tilde{\phi}_j - \phi) \to 0$ as $n \to \infty$

which implies from (2.5) that

$$E \int_0^t Q(f_s^*(\sum_{j=m}^{n} \langle\phi,\tilde{\phi}_j\rangle_{q(t,f)}\tilde{\phi}_j), f_s^*(\sum_{j=m}^{n} \langle\phi,\tilde{\phi}_j\rangle_{q(t,f)}\tilde{\phi}_j))ds \to 0$$

as $n.m \to \infty$. (2.9)

On the other hand, since $\psi_j[\phi] = \langle\phi,\tilde{\phi}_j\rangle_{q(t,f)}$ then

$$\tilde{Y}_t(f)[\phi] = \sum_{j=1}^{\infty} Y_t(f)[\tilde{\phi}_j]\psi_j[\phi] = \sum_{j=1}^{\infty} Y_t(f)[\tilde{\phi}_j]\langle\phi,\tilde{\phi}_j\rangle_{q(t,f)}$$

$$= \sum_{j=1}^{\infty} Y_t(f)[\langle\phi,\tilde{\phi}_j\rangle_{q(t,f)}\tilde{\phi}_j] \quad \text{a.s.} \quad .$$

Thus if $g_n(s) = f_s^*(\sum_{j=1}^{n} \langle\phi,\tilde{\phi}_j\rangle_{q(t,f)}\tilde{\phi}_j)$

$$\tilde{Y}_t(f)[\phi] = \lim_{n\to\infty} \int_0^t \langle g_n(s), dW_s\rangle_Q \qquad \text{a.s.}$$

and from (2.9) and Definition 2.3

$$\int_0^t \langle g_n(s), dW_s\rangle_Q \to \int_0^t \langle f_s^*(\phi), dW_s\rangle_Q \quad \text{in } L^2(\Omega).$$

Thus for each $t > 0$ $\tilde{Y}_t(f)[\phi] = \int_0^t \langle f_s^*(\phi), dW_s\rangle_Q$ a.s. $\forall\ \phi \in \Phi$.
From now on we write $Y_t(f)$ instead of $\tilde{Y}_t(f)$.

Q.E.D.

The Φ'-valued stochastic integral $Y_t(f) = \int_0^t f_s dW_s$ has the following properties.

Proposition 2.4 Let $f, g \in \mathcal{O}_Q(\Phi', \Phi')$.

a) If $a, b \in \mathbb{R}$ then for each $t > 0$

$$Y_t(af+bg) = aY_t(f) + bY_t(g) \quad \text{a.s.} \quad .$$

b) $E(Y_t(f)[\phi]) = 0 \quad \forall\, \phi \in \Phi \quad t > 0$.

c) $E(Y_t(f)[\phi] Y_t(f)[\psi]) = E \int_0^t Q(f_s^*(\phi), f_s^*(\psi))ds \quad \forall\, \phi, \psi \in \Phi.$

d) $E\| Y_t(f) \|^2_{-q(t,f)} = E \int_0^t \| f_s \|^2_{\sigma_2(H_Q, H_{-q(t,f)})} ds < \infty$

$\forall\, t > 0$, for some $q(t,f) > r_1 + r_2$.

We now extend the definition of $Y_t(\cdot)$ to functions which are integrable in $[0,\infty) \times \Omega$. Lemma 2.2 and Proposition 2.4 (d) suggest that it is enough to construct stochastic integrals for functions of the form $f:[0,\infty) \times \Omega \to \sigma_2(H_Q, H_{-r})$ for $r > 0$, similar to the case of a c.B.m. (see [21]).

Let $r > 0$. A function $f:[0,\infty) \times \Omega \to \sigma_2(H_q, H_{-r})$ is said to belong to the class $\mathcal{O}(H_Q, H_{-r})$ if f is an F_t-adapted measurable function on $\mathbb{R}_+ \times \Omega$ to $\sigma_2(H_Q, H_{-r})$ such that

$$\int_0^\infty E\| f_s \|^2_{\sigma_2(H_Q, H_{-r})} ds < \infty.$$

<u>Proposition 2.5</u> Let $r > r_1 + r_2$ and $f \in \mathcal{O}(H_Q, H_{-r})$. Then there exists an H_{-r}-valued element $Y(f)$, called the stochastic integral for elements in $\mathcal{O}(H_Q, H_{-r})$, such that

$$Y(f)[\phi] = \int_0^\infty \langle f_s^*(\phi), dW_s \rangle_Q \quad \text{a.s.} \quad \forall\, \phi \in H_r$$

where the RHS is the stochastic integral of Definition 2.3. We sometimes denote this integral by $Y(f) = \int_0^\infty f_s \, dW_s$.

It has the following properties: If $f, g \in \mathcal{O}(H_Q, H_{-r})$

a) For $a, b \in \mathbb{R}$ $\quad Y(af+bg) = aY(f) + bY(g) \quad$ a.s..

b) $E(Y(f)[\phi]) = 0 \qquad \forall\, \phi \in H_r$.

c) $E(Y(f)[\phi]\, Y(g)[\psi]) = E \int_0^\infty Q(f_s^*(\phi), g_s^*(\psi))\, ds \quad \phi, \psi \in H_r$.

d) $E\| Y(f) \|_{-r}^2 \quad \int_0^\infty \| f_s \|^2_{\sigma_2(H_Q, H_{-r})} ds < \infty$.

e) If $Y_t(f) = \int_0^t f_s \, dW_s$, then (Y_t, F_t^W) $t \geq 0$ is a Φ'-valued square integrable martingale with an H_{-r} continuous version, for $r > r_1 + r_2$.

3. MULTIPLE WIENER INTEGRALS

Real valued multiple Wiener integrals. Let $n > 1$, $q > r_1 + r_2$, $T = [0,\infty)$ and denote by $H_q^{\otimes n}$ the n-fold tensor product of H_q with itself. For $f \in L^2(T^n \to H_q^{\otimes n})$ define

$$I_n(f) = \sum_{j_1,\dots,j_n=1}^{\infty} \int_{T^n} \langle f(\underset{\sim}{t}), e_{j_1} \otimes \dots \otimes e_{j_n} \rangle_{q^{\otimes n}} \, dW_{t_1}[e_{j_1}] \dots dW_{t_n}[e_{j_n}]$$

$$\underset{\sim}{t} = (t_1,\dots,t_n) \tag{3.1}$$

where $[e_j]_{j \geq 1}$ is a CONS in H_q and each multiple integral in RHS of (3.1) is a multiple Wiener integral (m.W.i) with dependent integrators $(W_{t_1}[e_{j_1}],\dots,W_{t_n}[e_{j_n}])$ of the type considered in [3], [4] and [17].

Similar to the stochastic integral (2.1), and using the nuclearity of Φ and the properties of Q, it can be shown that the real valued multiple Wiener integral (3.1) is well defined and its value does not depend on the CONS $\{e_j\}_{j \geq 1}$ of H_q or the choice of q. Moreover $I_n(\cdot)$ is a linear operator and

$$E(I_n(f))^2 \leq n! \, \| f \|^2_{L^2(T^n \to H_Q^{\otimes n})}. \tag{3.2}$$

For $f \in L^2(T^n \to H_Q^{\otimes n})$ a m.W.i. of the form (3.1) cannot be defined since W_t is not an H_Q-valued process. However since

for all $q > r_1 + r_2$ $L^2(T^n \to H_q^{\otimes n})$ is dense in $L^2(T^n \to H_Q^{\otimes n})$ then by (3.2) $I_n(\cdot)$ has a unique extension to $L^2(T^n \to H_Q^{\otimes n})$ also denoted by I_n and called the <u>real valued multiple Wiener integral</u> for elements in $L^2(T^n \to H_Q^{\otimes n})$. It has the usual properties of the ordinary multiple Wiener integral of Itô [8]. In particular the following is true.

<u>Lemma 3.1</u>.- Let $f \in L^2(T^n \to H_Q^{\otimes n})$. Then there exists $g \in M_Q$, $E \int_0^\infty \| g(s) \|_Q^2 \, ds < \infty$ such that

$$I_n(f) = \int_0^\infty \langle g(s), dW_s \rangle_Q \tag{3.3}$$

where RHS of (3.3) is the stochastic integral of Definition 2.3

<u>Φ'-valued multiple Wiener integrals</u>.- Let $s \geq r_1 + r_2$ and $\sigma_2(H_Q^{\otimes n}, H_{-s})$ denote the Hilbert space of Hilbert-Schmidt operators from $H_Q^{\otimes n}$ to H_{-s}.

<u>Proposition 3.1</u> Let $f \in L^2(T^n \to \sigma_2(H_Q^{\otimes n}, H_{-s}))$. Then there exists an H_{-s}-valued element $Y_n(f)$ such that

$$Y_n(f)[\phi] = I_n(f^*(\phi)) \quad \text{a.s.} \quad \forall \phi \in H_s \tag{3.4}$$

where $I_n(\cdot)$ is the real valued m.W.i. defined above. $Y_n(f)$

is called the <u>Φ'-valued multiple Wiener integral</u> of f.
It is such that

$$E\|Y_n(f)\|^2_{-s} < n! \ \|f\|^2_{L^2(T^n \to \sigma^2(H_Q^{\otimes n}, H_{-s}))}. \qquad (3.5)$$

The following result is an infinite dimensional analog of Lemma 6.7.2 in Kallianpur [11].

<u>Proposition 3.2</u> Let $n > 1$, $s > r_1 + r_2$ and $f \in L^2(T^n \to \sigma_2(H_Q^{\otimes n}, H_{-s}))$. Then there exists a non-anticipative $\sigma_2(H_Q, H_{-s})$-valued process $h(t,\omega)$ such that

$$E\int_0^\infty \|h(t,\omega)\|^2_{\sigma_2(H_Q,H_{-s})} dt < \infty \qquad (3.6)$$

and

$$Y_n(f) = \int_0^\infty h_t \, dW_t \qquad (3.7)$$

where RHS in (3.7) is the Φ'-valued stochastic integral of Proposition 2.5.

<u>Proof</u> By Lemma 3.1 for each $\phi \in H_s$ there exists $g_\phi \in M_Q$, $E(\int_0^\infty \|g_\phi(s)\|^2_Q \, ds < \infty$ and

$$Y_n(f)[\phi] = \int_0^\infty \langle g_\phi(t), dW_t\rangle_Q = I_n(f^*(\phi)) \text{ a.s. } . \qquad (3.8)$$

Let $\{e_k\}_{k \geq 1}$ be a CONS for H_s and define $h^*(t,\omega)(e_k) = g_{e_k}(t,\omega)$ $k \geq 1$. Then $h^*(t)(e_k)$ is H_Q-valued and belongs to $\mathcal{M}_Q$. Next

$$\int_0^\infty E\Big(\sum_{k=1}^\infty \| h^*(t)(e_k) \|_Q^2\Big)dt = \sum_{k=1}^\infty E\Big(\int_0^\infty \langle g_{e_k}(t), dW_t\rangle_Q\Big)^2$$

$$= \sum_{k=1}^\infty E(Y_n(f)[e_k])^2 \leq n! \sum_{k=1}^\infty \| f^*(e_k) \|^2_{L^2(T^n \to H_Q^{\otimes n})}$$

$$= n! \; \| f \|^2_{L^2(T^n \to \sigma_2(H_Q,\, H_{-s}))} < \infty \quad .$$

Thus $h^*(t)(\cdot) = \sum_{k=1}^\infty \langle \cdot, e_k\rangle_s \; h^*(t)(e_k)$ defines an a.s.

dtdP linear operator from H_s to H_Q. Moreover $h^*(t,\omega) \in \sigma_2(H_Q, H_{-s})$ a.s. dtdP and

$$E \int_0^\infty \| h(t) \|^2_{\sigma_2(H_Q,\, H_{-s})} \, dt \quad .$$

Then the result follows from Proposition 2.5.

Q.E.D.

As in the case of a real valued Wiener process, in the next section we will see how the m.W.i. above defined are useful in studying Φ'-valued nonlinear functionals of the Φ'-valued Wiener process $W_\cdot$.

4. Φ'-VALUED NONLINEAR FUNCTIONALS

Let $F^W = F^W_\infty$. By a Φ'-valued nonlinear functional of W we mean a Φ'-valued random element $F: \Omega \to \Phi'$ such that F is $F^W \to \mathcal{B}(\Phi')$ measurable and $E(F[\phi])^2 < \infty \;\; \forall \phi \in \Phi$. We denote by $L^2(\Omega\,;\,\Phi')$ the linear space of all Φ'-valued nonlinear functionals. Observe that it is not a Hilbert space.

For $r > 0$ let $L^2(\Omega \to H_{-r})$ be the Hilbert space of all F^W-measurable elements $F: \Omega \to H_{-r}$ such that $E(\|F\|^2_{-r}) < \infty$. The Hilbert space $L^2(\Omega \to H_{-r})$ is called the space of H_{-r}-valued nonlinear functionals of W_t.

Let $H = \overline{sp}\,\{W_t[\phi]: \phi \in \Phi,\ t \in T\}$ (closure with respect to $L^2(\Omega, F^W, P)$) and $H^{\odot n}$ be its n-fold symmetric tensor product. Since H is a Gaussian space, it is well known that $L^2(\Omega, F^W, P) = \sum_{n \geq 0} \oplus H^{\odot n}$. For $s > 0$ and $n \geq 1$ define

$$G_n(H_{-s}) = \{\eta \in L^2(\Omega \to H_{-s}): \eta[\phi] \in H^{\odot n} \;\forall \phi \in H_s\}. \quad (4.1)$$

The following result is the Wiener decomposition of the space $L^2(\Omega\,;\,\Phi')$.

<u>Theorem 4.1</u> The linear space $L^2(\Omega\,;\,\Phi')$ of Φ'-valued nonlinear functionals is a (complete) nuclear space given by the strict inductive limit of the Hilbert spaces $L^2(\Omega \to H_{-r})_{r > 0}$.

Moreover

$$L^2(\Omega\ ;\ \Phi') = \varinjlim_{r\to\infty} \Big(\sum_{n\geq 0}{}^{\oplus}\ G_n(H_{-r}) \Big) .$$

The proof of this theorem is based on the following lemmas.

Lemma 4.1

$$L^2(\Omega\ ;\ \Phi') = \bigcup_{r=0}^{\infty} L^2(\Omega \to H_{-r}) . \tag{4.2}$$

Proof Let $F \in L^2(\Omega \to H_{-r})$ $r \geq 0$. Then $F[\phi]$ is F^W-measurable for all $\phi \in \Phi$ and $E(F[\phi])^2 \leq \|\phi\|_r^2\, E\|F\|_{-r}^2 < \infty$, i.e. $F \in L^2(\Omega \to \Phi')$ and hence

$$\bigcup_{r=0}^{\infty} L^2(\Omega \to H_{-r}) \subset L^2(\Omega\ ;\ \Phi') .$$

Next let $F \in L^2(\Omega\ ;\ \Phi')$ and for all $\phi \in \Phi$ define $V^2(\phi) = E(F[\phi])^2$. Then $V^2(\phi) < \infty$ $\forall\ \phi \in \Phi$.

As in the proof of Lemma 2.2 using the continuity of F on Φ and Fatou's lemma, one can show that $V(\phi)$ is a lower semi-continuous function of Φ. Then by Lemma I.2.3 in [20, page 386], $V(\phi)$ is a continuous function on Φ and hence there exist $\theta_F > 0$ and $r_F > 0$ such that

$$V^2(\phi) = E(F(\phi))^2 \leq \theta_F^2 \|\phi\|_{r_F}^2 \quad \forall \phi \in \Phi. \tag{4.3}$$

Let $r > r_F + r_1$, then the imbedding of H_r into H_{r_F} is a Hilbert-Schmidt map. Take $\tilde{\phi}_j = (1 + \lambda_j)^{-r}\phi_j$, then $\{\tilde{\phi}_j\}_{j \geq 1}$ is a CONS in H_r and

$$E(\sum_{j=1}^{\infty} F[\tilde{\phi}_j]^2) = \sum_{j=1}^{\infty} V^2(\tilde{\phi}_j)$$

$$\leq \theta_F^2 \sum_{j=1}^{\infty} \|\tilde{\phi}_j\|_{r_F}^2 = \theta_F^2 \sum_{j=1}^{\infty} (1 + \lambda_j)^{-2(r-r_F)} \leq \theta_F \theta_1 < \infty$$

where θ_1 is as in (1.2). Then $\Sigma_{j=1}^{\infty} F[\tilde{\phi}_j]^2 < \infty$ a.s., and if $\{\psi_j\}_{j \geq 1}$ is the CONS in H_{-r} dual to $\{\tilde{\phi}_j\}_{j \geq 1}$

$$P(\tilde{F}(\omega) = \sum_{j=1}^{\infty} F[\tilde{\phi}_j](\omega)\psi_j < \infty) = 1$$

and $\tilde{F} \in H_{-r}$ a.s. . Moreover,

$$E\|\tilde{F}\|_{-r}^2 = \sum_{j=1}^{\infty} E\langle\tilde{F}, (1+\lambda_j)^r\phi_j\rangle_{-r}^2 = E(\sum_{j=1}^{\infty} F[\tilde{\phi}_j]^2) < \infty.$$

It remains to show that for each $\phi \in \Phi$ $F[\phi] = \tilde{F}[\phi]$. By using (4.3) since $\Sigma_{j=1}^{m} \langle\phi, \tilde{\phi}_j\rangle_r \tilde{\phi}_j \underset{m\to\infty}{\to} \phi$ in H_r

$$E(F[\phi] - \sum_{j=1}^{m} F[\tilde{\phi}_j]\psi_j[\phi])^2 = E(F[\phi - \sum_{j=1}^{m} \langle\phi, \tilde{\phi}_j\rangle_r \tilde{\phi}_j])^2$$

$$\leq \theta_F^2 \|\phi - \sum_{j=1}^{m} \langle\phi, \tilde{\phi}_j\rangle_r \tilde{\phi}_j\|_r^2 \to 0 \quad \text{as} \quad m \to \infty$$

and therefore for each $\phi \in \Phi$ $\tilde{F}[\phi] = F[\phi]$ a.s.. Thus if $F \in L^2(\Omega ; \Phi')$ there exists $r > 0$ such that $F \in L^2(\Omega \to H_{-r})$.

Q.E.D.

The next result is the Wiener decomposition of the space $L^2(\Omega \to H_{-r})$. It appears in Miyahara [16] for a general Hilbert space K, i.e. for $L^2(\Omega \to K)$.

<u>Lemma 4.2</u> For each $r > 0$ $L^2(\Omega \to H_{-r}) = \sum_{n \geq 0}^{\oplus} G_n(H_{-r})$.

<u>Proof of Theorem 4.1</u> It follows by Lemmas 4.1 and 4.2 and the fact that $L^2(\Omega ; \Phi')$ is the dual of the Countably Hilbert Nuclear space $\bigcap_{r \geq 0} L^2(\Omega \to H_r)$.

Q.E.D.

Define for $n > 1$ $G_n(\Phi') = \{\eta \in L^2(\Omega ; \Phi'):\eta[\phi] \in H^{\odot n} \ \forall \phi \in \Phi\}$. The following corollary is shown similar to Theorem 4.1.

<u>Corollary 4.1</u> For each $n > 1$ $G_n(\Phi')$ is a (complete) nuclear space given by the strict inductive limit of the Hilbert spaces $G_n(H_{-r})$ $r > 0$.

<u>Multiple Wiener integral orthogonal expansions</u>.- Let

$$S_Y = \{Y_n(f_n):f_n \in L^2(T^n \to \sigma_2(H_Q^{\otimes n},H_{-s})) \ n > 0, \ s > 0\}$$

where $Y_n(\cdot)$ is the m.W.i. of Proposition 3.1. We shall show that S_Y is a complete set in the space $L^2(\Omega;\ \Phi')$.

For $r > 0$ let S_Y^r be the closed subspace of $L^2(\Omega \to H_{-r})$ spanned by the multiple Wiener integrals $Y_n(\cdot)$ of Proposition 3.1 for elements in $L^2(T^n \to \sigma_2(H_Q^{\otimes n}, H_{-r}))$, i.e.

$$S_Y^r = \overline{\mathrm{sp}}\ \{Y_n(f_n):\ f_n \in L^2(T^n \to \sigma_2(H_Q^{\otimes n}, H_{-r}))\ n \geq 1\}$$

where the closure is taken with respect to $L^2(\Omega \to H_{-r})$.

Although multiple Wiener integrals on a Hilbert space have been studied before (Miyahara [16]), the following result was not found in the literature.

<u>Proposition 4 1</u> For each $r > 0$

$$\sigma_2(\mathrm{EXP}(L^2(T) \otimes H_Q),\ H_{-r}) \overset{\xi}{\cong} S_Y^r$$

where for $g \in \sigma_2(\mathrm{EXP}(L^2(T) \otimes H_Q),\ H_{-r})$, $g^* = (g_0^*, g_1^*, \ldots)$

$$g_n^* \in \sigma_2(H_r, (L^2(T) \otimes H_Q)^{\odot n}) \quad n > 0$$

$$\xi(g) = \sum_{n=0}^{\infty} Y_n(g_n) \qquad \text{(convergence in } L^2(\Omega \to H_{-r}))$$

where $\mathrm{EXP}(L^2(T) \otimes H_Q) = \sum_{n \geq 0} \oplus\ (L^2(T) \otimes H_Q)^{\odot n}$.

Proof Let $g \in \sigma_2(EXP(L^2(T) \otimes H_Q), H_{-r})$, then $g^* \in \sigma_2(H_r, EXP(L^2(T) \otimes H_Q))$, i.e. for each $\phi \in H_s$ $g^*(\phi) \in EXP(L^2(T) \otimes H_Q)$, $g^*(\phi) = (g_o^*(\phi), g_1^*(\phi), \ldots)$ and

$$\sum_{n=0}^{\infty} \| g_n^*(\phi) \|^2_{(L^2(T) \otimes H_Q)^{\odot n}} < \infty .$$

We first show that for each $n \geq 0$ $g_n^* \in \sigma_2(H_r, (L^2(T) \otimes H_Q)^{\odot n})$. Let $\{e_m\}_{m \geq 1}$ be a CONS in H_r, then

$$\sum_{m=1}^{\infty} \| g^*(e_m) \|^2_{EXP(L^2(T) \otimes H_Q)} < \infty$$

and hence

$$\sum_{m=1}^{\infty} \| g^*(e_m) \|^2_{EXP(L^2(T) \otimes H_Q)}$$

$$= \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} \| g_n^*(e_m) \|^2_{(L^2(T) \otimes H_Q)^{\odot n}} < \infty.$$

Thus for each n and $\{e_m\}_{m \geq 1}$ a CONS for H_r

$$\sum_{m=1}^{\infty} \| g_n^*(e_m) \|^2_{(L^2(T) \otimes H_Q)^{\odot n}} < \infty,$$

i.e. $g_n^* \in \sigma_2(H_r, (L^2(T) \otimes H_Q)^{\odot n})$ $n \geq 0$.

But if $g \in \sigma_2(EXP(L^2(T) \otimes H_Q), H_{-r})$

$$E\|\xi(g)\|^2_{-r} = \sum_{n=0}^{\infty} E\|Y_n(g_n)\|^2_{-r} = \|g\|^2_{\sigma_2(EXP(L^2(\mathbb{R}_+)\otimes H_Q),H_{-r})}.$$

Then the result follows since g as above is a typical element in $\sigma_2(EXP(L^2(T) \otimes H_Q), H_{-r})$ $r > 0$.

Q.E.D.

The completeness of the multiple Wiener integrals $Y_n(f_n)$, $f_n \in L^2(T^n \to \sigma_2(H_Q^{\otimes n}, H_{-r}))$ in $L^2(\Omega \to H_{-r})$ is then obtained.

<u>Proposition 4.2</u> Let $r \geq 0$ and $F \in L^2(\Omega \to H_{-r})$, $E(F) = \underline{0}$. Then

$$F = \sum_{n=1}^{\infty} Y_n(f_n) \quad \text{a.s.} \quad (\text{convergence in } L^2(\Omega \to H_{-r}))$$

where $f_n \in L^2(T^n \to \sigma_2\ (H_Q^{\otimes n}, H_{-r}))$ $n \geq 1$.

The above proposition and Theorem 4.1 yield the next result which gives multiple Wiener integral expansions for Φ'-valued nonlinear functionals.

<u>Theorem 4.2</u> Let $F \in L^2(\Omega\ ; \Phi')$, $E(F[\phi]) = 0$ $\forall \phi \in \Phi$. Then there exists $r_F > 0$ such that $F \in H_{r_F}$ a.s. and

$$F = \sum_{n=1}^{\infty} Y_n(f_n) \quad \text{a.s.} \quad (L^2(\Omega \to H_{-r_F})\text{-convergence})$$

where $f_n \in L^2(T^n \to \sigma_2(H_Q^{\otimes n}, H_{-r_F}))$ $n \geq 1$.

<u>Stochastic integral representations for Φ'-valued nonlinear functionals</u> From Proposition 3.2 and Theorem 4.2 one obtains the following stochastic integral representation for elements in $L^2(\Omega\,;\,\Phi')$. This result is the Φ'-valued analog of Theorem 6.7.1 in Kallianpur [11].

<u>Theorem 4.3</u> Let $F \in L(\Omega\,;\,\Phi')$, $E(F[\phi]) = 0 \quad \forall\, \phi \in \Phi$. Then there exist $r_F > 0$ and a non-anticipative $\sigma_2(H_Q, H_{-r_F})$-valued process h with

$$\int_0^\infty E\| h(t,\omega) \|^2_{\sigma_2(H_Q, H_{-r_F})} dt < \infty$$

such that

$$F(\omega) = \int_0^\infty h(t,\omega)\, dW_t \qquad \text{a.s.}$$

where the RHS in the last expression is the Φ'-valued stochastic integral of Proposition 2.5 with an H_{-r_F}-valued continuous version.

The last theorem and an application of the Baire category theorem (as in Theorem 4.1) yield the following representation theorem for Φ'-valued square integrable martingales (see [17] for details). A Φ'-valued stochastic process $(X_t)_{t \geq 0}$ is said to be a Φ'-valued square integrable martingale with respect to an increasing family $(F_t)_{t \geq 0}$ of σ-fields if for

each $\phi \in \Phi$ $(X_t[\phi], F_t)_{t \geq 0}$ is a real valued square integrable martingale, i.e.

$$\sup_{0 \leq t < \infty} E(X_t^2[\phi]) < \infty \quad \forall \phi \in \Phi .$$

Theorem 4.4 Let (X_t, F_t^W), $X_o = 0$, be a Φ'-valued square integrable martingale. Then there exists $r_X > 0$ such that X_t has an H_{-r_X} continuous version $\tilde{X}_t$ given by the Φ'-valued stochastic integral

$$\tilde{X}_t(\omega) = \int_o^t h(s,\omega) d\, W_s \quad \text{a.s.} \tag{4.4}$$

for every $t \geq 0$, where $h(t,\omega)$ is nonanticipative, $\sigma_2(H_Q, H_{-r_X})$ - valued and

$$\int_o^\infty E\| h(t,\omega) \|^2_{\sigma_2(H_Q, H_{-r_X})} dt < \infty ,$$

where RHS of (4.4) is the Φ'-valued stochastic integral of Proposition 2.5.

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